

Avoiding closed timelike curves with a collapsing rotating null dust shell

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Abstract

We present an idealised model of gravitational collapse, describing a collapsing rotating cylindrical shell of null dust in flat space, with the metric of a spinning cosmic string as the exterior. We find that the shell bounces before closed timelike curves can be formed. Our results also suggest slightly different definitions for the mass and angular momentum of the string.

Introduction

A stationary, cylindrically-symmetric rotating solution of the Einstein equations containing closed timelike curves (CTCs) was first published in [18]. Its possible role as a time-machine was discussed by Tipler [16] and Bonnor [1]. Other metrics with CTCs include those of Gödel [7], Newman-Unti-Tamburino [13, 12], Gott [8] and the rotating cosmic string [5]. In [2, 3] Bonnor has emphasised the need for a proper understanding for the formation of CTCs.

Several attempts have been made to rule out the possibility of creating CTCs [17, 9, 6]. This would follow from a proof of the strong cosmic censorship conjecture, in the following sense: the domain of dependence of an appropriate¹ initial surface, evolving under the Einstein equations with a reasonable matter content, is (by construction) globally hyperbolic, and hence contains no CTCs. If in addition it is (generically) inextendible, which is the usual formulation of strong cosmic censorship, then there are no CTCs at all. It would then follow that, if one took an initial surface with initial data set up so that CTCs *could* form, in fact they would not.

It would be desirable to have non-stationary models which could provide test beds for this idea. In this paper we present a simplified example, obtained by matching a Minkowski interior across a collapsing, rotating cylindrical shell of null dust with positive energy density to a spinning cosmic string exterior². We find that the shell bounces before the critical radius at which CTCs would be formed can be reached.

We follow the conventions of [19]. The solutions considered are four-dimensional but the z -coordinate is ignorable, and we will omit it from the calculations.

1 Interior

We take the flat Minkowski interior,

$$g^- = -d\tau^2 + d\xi^2 + d\eta^2,$$

¹The simplest condition is to take it compact.

²A similar matching was done across a stationary cylindrical shell in [11]; thin shell collapse in 2+1 dimensions was studied in [14], [4].

which we shall match to the spinning cosmic string exterior along the hyperboloid

$$-\tau^2 + \xi^2 + \eta^2 = a^2. \quad (1)$$

As is well known, this hyperboloid is ruled by two families of null geodesics. We choose coordinates (u, ψ) on the hyperboloid adapted to one of these families, given by

$$\begin{cases} \tau = u \\ \xi = a \cos \psi - u \sin \psi \\ \eta = a \sin \psi + u \cos \psi \end{cases}$$

The induced metric on the hyperboloid is given in these coordinates by

$$h^- = 2a du d\psi + (u^2 + a^2) d\psi^2. \quad (2)$$

The second fundamental form of this hyperboloid is known to be proportional to the metric, a fact we obtain below by an indirect route.

2 Exterior

The metric for a spinning cosmic string is [5, 11]

$$g^+ = -(dt + m d\varphi)^2 + C^2 dr^2 + r^2 d\varphi^2,$$

with $m > 0$, $C > 0$. Notice that since $g_{\varphi\varphi}^+ = r^2 - m^2$, the trajectories of the $\partial/\partial\varphi$ -Killing vector define closed null curves if $r = m$ and CTCs if $r < m$. The parameters C and m are conventionally related [11] to the mass per unit length μ and angular momentum per unit length J of the string through

$$\mu = \frac{C-1}{4C}; \quad (3)$$

$$J = \frac{m}{4}, \quad (4)$$

so that for positive mass $C > 1$. Our calculation, based on Killing vectors in the Minkowski space interior to the shell, suggests a slightly different identification of mass and angular momentum per unit length in terms of C and m , and the use of the Killing vectors in the exterior would give again a slightly different identification³.

We must select a timelike surface in this spacetime which is ruled by a family of null geodesics, to be the matching surface. The geodesic Lagrangian is

$$L = \frac{1}{2} [-(\dot{t} + m\dot{\varphi})^2 + C^2\dot{r}^2 + r^2\dot{\varphi}^2],$$

and the equations for null geodesics are

$$\frac{\partial L}{\partial \dot{t}} = -E \quad \Leftrightarrow \quad \dot{t} + m\dot{\varphi} = E; \quad (5)$$

$$\frac{\partial L}{\partial \dot{\varphi}} = K \quad \Leftrightarrow \quad r^2\dot{\varphi} = K + mE; \quad (6)$$

$$L = 0 \quad \Leftrightarrow \quad C^2 r^2 \dot{r}^2 = E^2 r^2 - (K + mE)^2. \quad (7)$$

³The stationary observers on the exterior can be shown to be moving with speed $v = \frac{C^2-1}{C^2+1}$ with respect to the stationary observers on the interior; see [15] for a similar discussion regarding collapsing spherical null shells.

for constants E and K . From (7) it is clear that if $\dot{\varphi} \neq 0$ then r always has a turning point, where it reaches its minimum value, which is

$$b = \frac{K}{E} + m,$$

where we assume that the geodesic is future-directed ($E > 0$) and rotating in the positive direction ($\dot{\varphi} > 0$). We use this to introduce the parameter λ through

$$r = b \sec \lambda, \tag{8}$$

which, by (7), satisfies

$$\dot{\lambda} = \frac{E \cos^2 \lambda}{bC},$$

choosing the positive root. Equations (6) and (5) then yield

$$\begin{cases} \varphi = C\lambda + \psi \\ t = bC \tan \lambda - mC\lambda \end{cases}$$

We may regard these equations, together with (8), as a transformation to a new set of coordinates $\{\lambda, b, \psi\}$, for which the metric becomes

$$g^+ = 2bC(b-m) \sec^2 \lambda d\lambda d\psi + C^2 db^2 - 2mC \tan \lambda db d\psi + (b^2 \sec^2 \lambda - m^2) d\psi^2. \tag{9}$$

The surfaces of constant b are ruled by a family of null geodesics, as desired; we will now match one of these surfaces to the hyperboloid (1).

2.1 The matching surface

On the exterior, the metric induced on a hypersurface $\{b = \text{constant}\}$ is

$$h^+ = 2bC(b-m) \sec^2 \lambda d\lambda d\psi + (b^2 \sec^2 \lambda - m^2) d\psi^2.$$

Introducing the coordinate

$$u = b \tan \lambda$$

this becomes

$$h^+ = 2C(b-m) du d\psi + (u^2 + b^2 - m^2) d\psi^2. \tag{10}$$

Comparing (10) with (2), we see that the metric on the two surfaces matches if

$$\begin{cases} a = C(b-m) \\ a^2 = b^2 - m^2 \end{cases} \quad \text{or equivalently} \quad \begin{cases} a = \frac{2C}{C^2-1}m \\ b = \frac{C^2+1}{C^2-1}m \end{cases} \tag{11}$$

Note in particular that the matching requires $b > m$, so that the shell bounces before closed causal curves are revealed in the exterior. Thus the spacetime has the property claimed in the introduction, provided it has a physically reasonable, distributional matter content on the shell.

2.2 Second fundamental form

To compute the distributional energy-momentum tensor of the shell, we next need to compute the second fundamental form of the matching surface from the two sides. From the exterior, this is the second fundamental form of the hypersurface $\{b = \text{constant}\}$. Note from (9) that a unit normal co-vector is $n = n_\alpha dx^\alpha = Cdb$. Therefore

$$n^\alpha \partial_\alpha = \frac{m \sin \lambda \cos \lambda}{bC(b-m)} \frac{\partial}{\partial \lambda} + \frac{1}{C} \frac{\partial}{\partial b}.$$

Using

$$\mathcal{L}_n d\lambda = d(\iota(n)d\lambda) + \iota(n)d(d\lambda) = \frac{m(\cos^2 \lambda - \sin^2 \lambda)}{bC(b-m)} d\lambda + (\dots) db$$

(where ι stands for contraction and the coefficient of db doesn't contribute to the second fundamental form), and similarly

$$\mathcal{L}_n db = \mathcal{L}_n d\psi = 0,$$

we find

$$\begin{aligned} K^+ &= \frac{1}{2}(\mathcal{L}_n g^+)_{|b=\text{constant}} = 2b \sec^2 \lambda d\lambda d\psi + \frac{b}{C} \left(\sec^2 \lambda + \frac{m}{b-m} \tan^2 \lambda \right) d\psi^2 \\ &= 2 du d\psi + \left(\frac{u^2}{C(b-m)} + \frac{b}{C} \right) d\psi^2. \end{aligned} \quad (12)$$

3 Consequences of Matching

The second fundamental form of the matching surface from the interior can be most easily obtained from (12): setting $C = 1$, $m = 0$ and $b = a$ we have

$$K^- = 2 du d\psi + \left(\frac{u^2}{a} + a \right) d\psi^2 = \frac{1}{a} h^-.$$

(As noted above, this is to be expected for a hyperboloid). The jump in the second fundamental form is therefore

$$\kappa = K^+ - K^- = -\frac{m}{C} d\psi^2,$$

where we have used the matching conditions (11). The Darmois-Israel formalism [10] yields the stress-energy tensor

$$T = S\delta(s),$$

where δ is a Dirac delta-function, s is the proper length along the spacelike geodesics orthogonal to the matching surface and

$$S = -\frac{1}{8\pi}(\kappa - (\text{tr } \kappa)h)$$

where $h = h^- = h^+$. It is easily seen that $\text{tr } \kappa = 0$, and hence

$$S = \frac{m}{8\pi C} d\psi^2.$$

Setting $l_\alpha dx^\alpha = d\psi$, we have $l^\alpha \partial_\alpha = \frac{1}{a} \frac{\partial}{\partial u}$. The contravariant version of S is then

$$S^{\alpha\beta} \partial_\alpha \otimes \partial_\beta = \frac{m}{8\pi C a^2} \frac{\partial}{\partial u} \otimes \frac{\partial}{\partial u},$$

implying that the matter content of the shell is a null dust with positive energy density moving along the null geodesics which rule the matching surface. Note that u is an affine parameter and $d\tau/du = 1$ along these geodesics.

In the interior, we have

$$s = a - \sqrt{\rho^2 - \tau^2},$$

where $\rho^2 = \xi^2 + \eta^2$. Using the identity

$$\delta(f(\rho)) = \frac{\delta(\rho - \rho_0)}{|f'(\rho_0)|}$$

for functions f with a single simple zero at $\rho = \rho_0$, we find

$$\delta(s) = \delta\left(\sqrt{\rho^2 - \tau^2} - a\right) = \frac{\sqrt{\rho^2 - \tau^2}}{\rho} \delta\left(\rho - \sqrt{\tau^2 + a^2}\right) = \frac{a}{\rho} \delta\left(\rho - \sqrt{\tau^2 + a^2}\right).$$

Therefore

$$T^{\alpha\beta} \partial_\alpha \otimes \partial_\beta = \frac{m}{8\pi C a \rho} \delta\left(\rho - \sqrt{\tau^2 + a^2}\right) \frac{\partial}{\partial u} \otimes \frac{\partial}{\partial u}.$$

Note that from (11) and (3) the surface energy density σ can be written as

$$\sigma = \frac{m}{8\pi a C \rho} = \frac{C^2 - 1}{16\pi C^2 \rho} = \frac{(C + 1)}{2C} \frac{\mu}{2\pi \rho}. \quad (13)$$

The conservation equation for the null dust composing the shell is equivalent to the constancy of $2\pi\rho\sigma$, which is true by (13), and then this quantity can be interpreted as the mass per unit length (in the z -direction). This suggests the identification of

$$\hat{\mu} = \frac{C^2 - 1}{8C^2} = \frac{(C + 1)}{2C} \mu$$

as the definition of mass per unit length, rather than μ . Note that $\hat{\mu}/\mu \rightarrow 1$ as $C \rightarrow 1$.

In cylindrical coordinates $\{\tau, \rho, \varphi\}$ for the interior, we have

$$\frac{\partial}{\partial u} = \frac{\partial}{\partial \tau} + A \frac{\partial}{\partial \rho} + B \frac{\partial}{\partial \varphi},$$

where A and B can be obtained from the conditions that $\partial/\partial u$ should be null and orthogonal to the hyperboloid. One finds

$$\frac{\partial}{\partial u} = \frac{\partial}{\partial \tau} + \frac{\tau}{\rho} \frac{\partial}{\partial \rho} + \frac{a}{\rho^2} \frac{\partial}{\partial \varphi},$$

and hence

$$-T^{\alpha\beta} \left(\frac{\partial}{\partial \tau}\right)_\alpha \left(\frac{\partial}{\partial \varphi}\right)_\beta = \rho^2 T^{\tau\varphi} = a\sigma \delta\left(\rho - \sqrt{\tau^2 + a^2}\right).$$

Therefore from (13) and (4) the surface angular momentum density j is

$$j = a\sigma = \frac{m}{8\pi C \rho} = \frac{1}{C} \frac{J}{2\pi \rho}.$$

This suggests the identification

$$\hat{J} = \frac{m}{4C} = \frac{J}{C}$$

as the definition of the string's angular momentum per unit length, rather than J . Again, $\hat{J}/J \rightarrow 1$ as $C \rightarrow 1$.

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